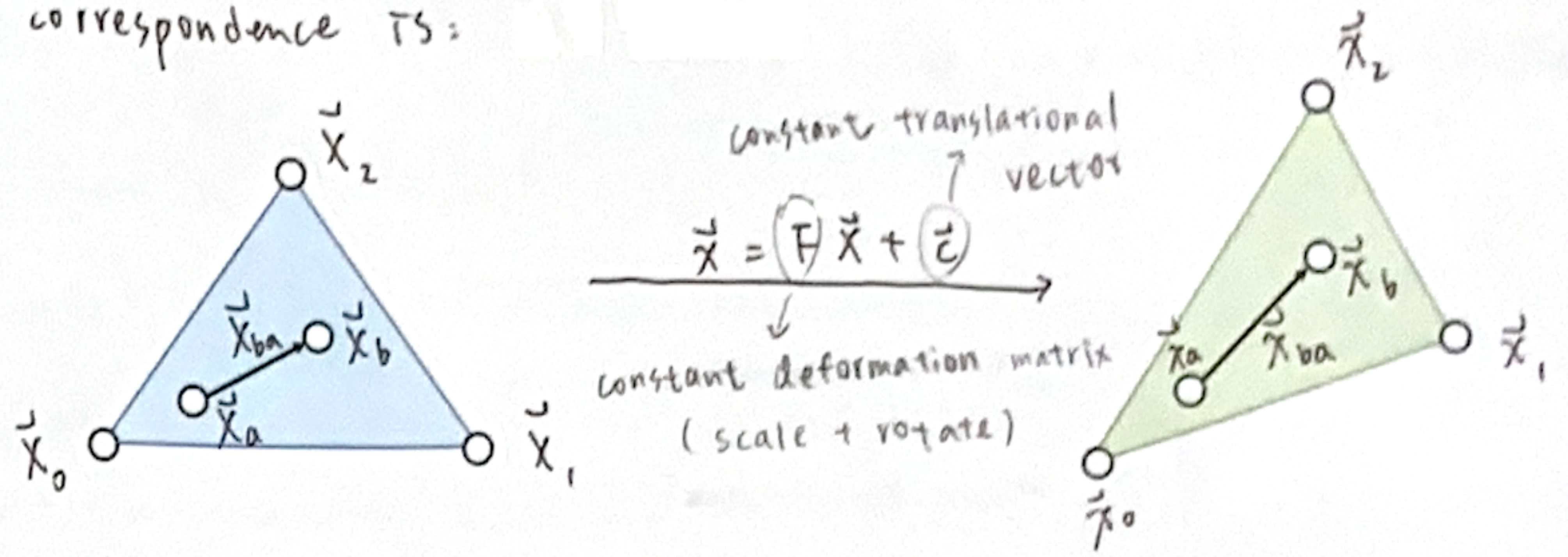
SB1

- Linear Finite Element Method (FEM)

Assume that for any point $\vec{X}$ in the reference triangle, its deformed correspondence is:



Reference (undeformed) Current (deformed)

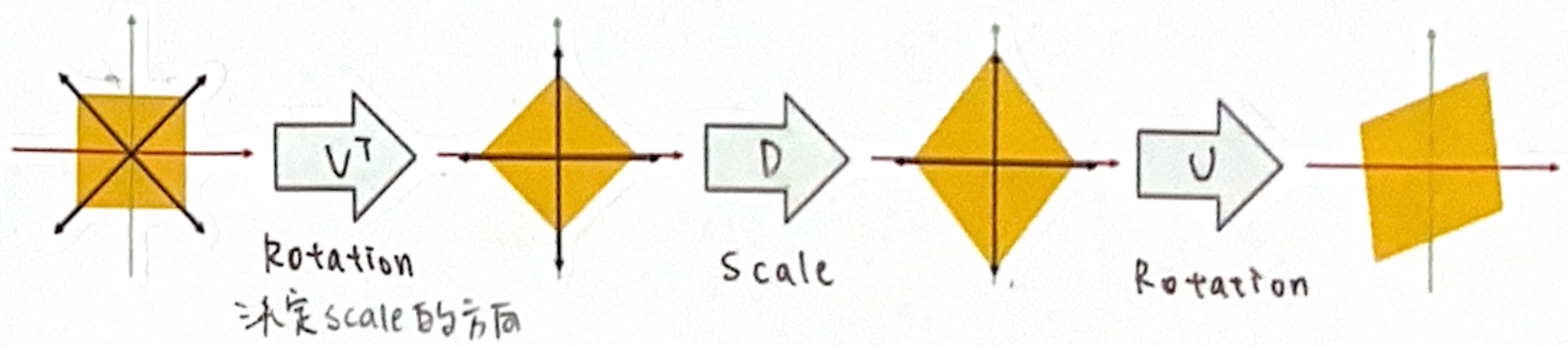
For any vector between two points, we can use F to convert it from reference to deformed:

$$\vec{x}_{ba} = \vec{x}_b - \vec{x}_a = F\vec{X}_b + \vec{c} - F\vec{X}_a - \vec{c} = F\vec{X}_b - F\vec{X}_a = F\vec{X}_{ba}$$

Therefore, calculate the deformation ^{F} gradient by edge vectors:

$$\begin{cases} F\vec{X}_{10} = \vec{x}_{10} \\ F\vec{X}_{20} = \vec{x}_{20} \end{cases} \rightarrow F[\vec{X}_{10}\ \vec{X}_{20}] = [\vec{x}_{10}\ \vec{x}_{20}] \rightarrow F = [\vec{x}_{10}\ \vec{x}_{20}][\vec{X}_{10}\ \vec{X}_{20}]^{-1}$$

Ideally, we need a tensor to describe shape deformation ONLY! Doing SVD on F gives $F = UDV^T$, where only V^T and D are relevant to deformation.



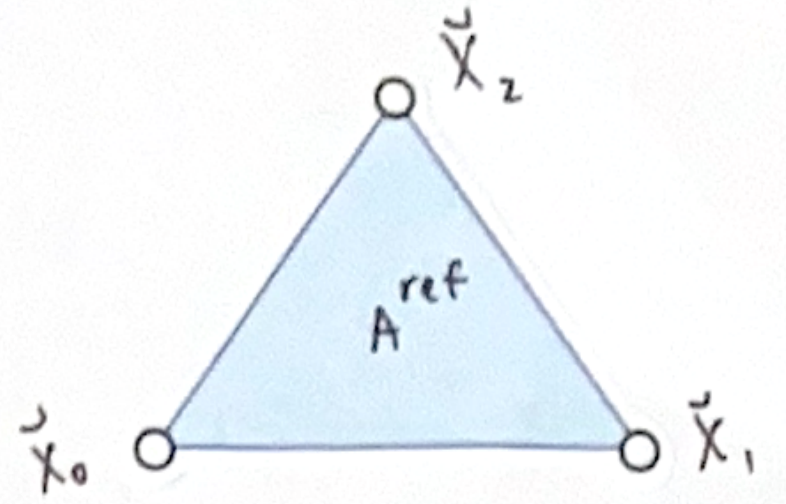
To get rid of U, define $G = \frac{1}{2}(F^TF - I) = \frac{1}{2}(VD^2V^T - I) = \begin{bmatrix} \varepsilon_{uu} & \varepsilon_{uv} \\ \varepsilon_{uv} & \varepsilon_{vv} \end{bmatrix}$

as Green Strain.

- If no deformation, $G = 0$; if deformation increase, $\|G\|$ increase because of $\|D\|$ increase.
- Three deformation modes: ε_{uu}, ε_{uv}, ε_{vv}
- G is rotation invariant. prove: $G = \frac{1}{2}(F^TR^TRF - I) = \frac{1}{2}(F^TF - I)$ R is the additional rotation

★ Strain Energy Density Function

G is the green strain describing deformation. Consider the energy density per reference area as $W(G)$.



- Total energy: $E = \int W(G)\, dA = A^{ref} W(\varepsilon_{uu}, \varepsilon_{vv}, \varepsilon_{uv})$ — constant with Δ
- By the Saint Venant–Kirchhoff Model (StVK):

$$W(\varepsilon_{uu}, \varepsilon_{vv}, \varepsilon_{uv}) = \frac{\lambda}{2}(\varepsilon_{uu}+\varepsilon_{vv})^2 + \mu(\varepsilon_{uu}^2+\varepsilon_{vv}^2+2\varepsilon_{uv}^2)$$

where λ and μ are Lamé parameters.

$$\rightarrow \frac{\partial W}{\partial G} = \begin{bmatrix} \frac{\partial W}{\partial \varepsilon_{uu}} & \frac{1}{2}\frac{\partial W}{\partial \varepsilon_{uv}} \\ \frac{1}{2}\frac{\partial W}{\partial \varepsilon_{uv}} & \frac{\partial W}{\partial \varepsilon_{vv}} \end{bmatrix} = 2\mu G + \lambda\, \mathrm{trace}(G)\, I = \underset{\sim}{S}$$

Second Piola-Kirchhoff tress tensor, something about force

★ Force

$$\vec{f}_i = -\left(\frac{\partial E}{\partial \vec{x}_i}\right)^T = -A^{ref}\left(\frac{\partial W}{\partial \vec{x}_i}\right)^T = -A^{ref}\left(\frac{\partial W}{\partial \varepsilon_{uu}}\frac{\partial \varepsilon_{uu}}{\partial \vec{x}_i} + \frac{\partial W}{\partial \varepsilon_{vv}}\frac{\partial \varepsilon_{vv}}{\partial \vec{x}_i} + \frac{\partial W}{\partial \varepsilon_{uv}}\frac{\partial \varepsilon_{uv}}{\partial \vec{x}_i}\right)^T$$

ex. (ref)

$$\vec{f}_1 = -A^{ref}\left(\frac{\partial W}{\partial \varepsilon_{uu}} a(a\vec{x}_{10}+c\vec{x}_{20})^T + \frac{\partial W}{\partial \varepsilon_{vv}} b(b\vec{x}_{10}+d\vec{x}_{20})^T + \frac{1}{2}\frac{\partial W}{\partial \varepsilon_{uv}} a(b\vec{x}_{10}+d\vec{x}_{20})^T + \frac{1}{2}\frac{\partial W}{\partial \varepsilon_{uv}} b(a\vec{x}_{10}+c\vec{x}_{20})^T\right)$$

$$= -A^{ref}\left(\begin{bmatrix} a(a\vec{x}_{10}+c\vec{x}_{20}) & b(b\vec{x}_{10}+d\vec{x}_{20}) \end{bmatrix}\begin{bmatrix} \frac{\partial W}{\partial \varepsilon_{uu}} a + \frac{1}{2}\frac{\partial W}{\partial \varepsilon_{uv}} b \\ \frac{1}{2}\frac{\partial W}{\partial \varepsilon_{uv}} a + \frac{\partial W}{\partial \varepsilon_{vv}} b \end{bmatrix}\right)$$

$$= -A^{ref} F \begin{bmatrix} \frac{\partial W}{\partial \varepsilon_{uu}} & \frac{1}{2}\frac{\partial W}{\partial \varepsilon_{uv}} \\ \frac{1}{2}\frac{\partial W}{\partial \varepsilon_{uv}} & \frac{\partial W}{\partial \varepsilon_{vv}} \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = -A^{ref} F S \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\vec{f}_2 = -A^{ref} F S \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\vec{f}_3 = -\vec{f}_1 - \vec{f}_2$$

$$\rightarrow [\vec{f}_1 \ \ \vec{f}_2] = -A^{ref} F S [\vec{X}_{10} \ \ \vec{X}_{20}]^{-T}$$

ref

Simplify by $[\vec{X}_{10}\ \vec{X}_{20}]^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\rightarrow F = [a\vec{x}_{10} + c\vec{x}_{20} \quad b\vec{x}_{10} + d\vec{x}_{20}]$

$\rightarrow G = \cdots$

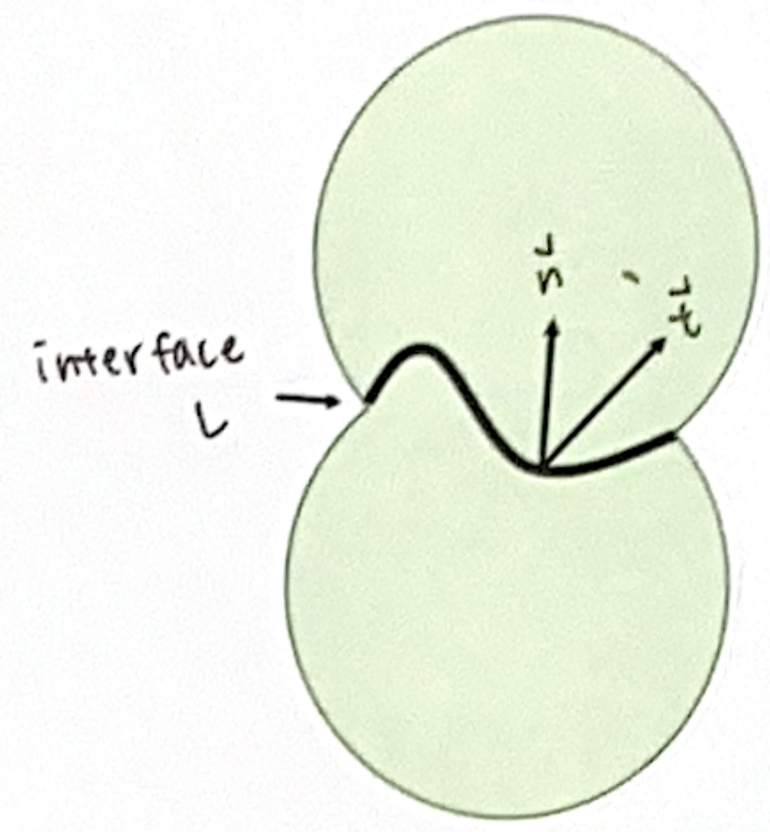
$\rightarrow \frac{\partial \varepsilon_{uu}}{\partial \vec{x}_1} = \cdots, \ \frac{\partial \varepsilon_{uv}}{\partial \vec{x}_1} = \cdots, \ \frac{\partial \varepsilon_{vv}}{\partial \vec{x}_1} = \cdots$

$\frac{\partial \varepsilon_{uu}}{\partial \vec{x}_2} = \cdots, \ \frac{\partial \varepsilon_{uv}}{\partial \vec{x}_2} = \cdots, \ \frac{\partial \varepsilon_{vv}}{\partial \vec{x}_2} = \cdots$

▲ Expand to tetrahedron (3D reference → 3D deformation)

- Same idea, but everything is now in 3D
- Deformation gradient $F \in R^{3\times3}$
- Green Strain $G \in R^{3\times3}$
- Stress tensor $S \in R^{3\times3}$
- Forces $\vec{f}_i \in R^3$

• Finite Volume Method (FVM)

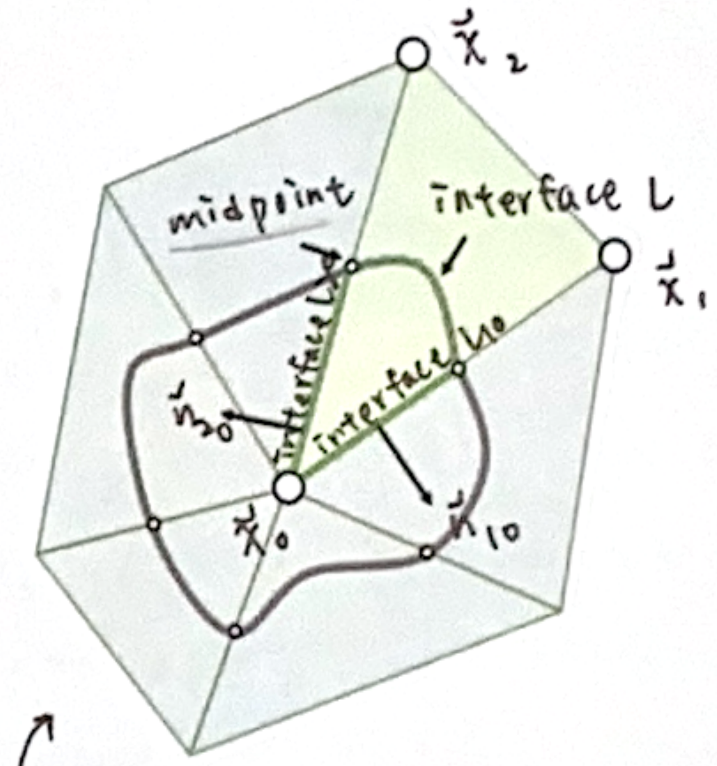


There is a traction force $\vec{t}$ that acting on a surface. Define $\vec{t}$ as the internal force per unit length (area in 3D).

- Total interface force: $\vec{f} = \oint_L \vec{t}\, dl$ $\rightarrow \vec{f} = \oint_L \sigma \vec{n}\, dl$
- Stress tensor σ: $\vec{t} = \sigma \vec{n}$

▲ Finite Volume Method considers force calculation in an integration perspective, not differentiation.

將面上的貢獻均勻分配到點上



In 2D plane

The force acting on $\vec{x}_0$ is contributed by the loop around it (that across all the midpoint).

→ force contributed by an element (green Δ):

$$\vec{f}_0 = \oint_L \sigma \vec{n}\, dl$$

每個mesh有不同的σ，但mesh上的σ都一樣

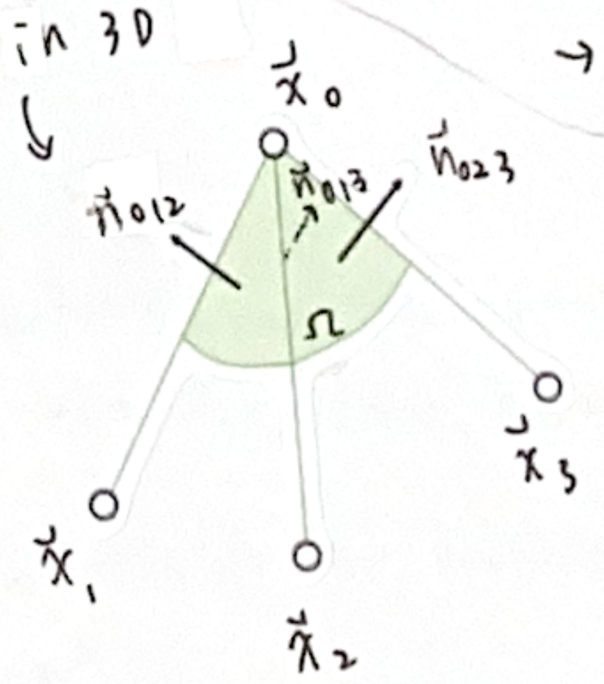
Since σ is a constant within the element,

$$\oint_L \sigma\vec{n}\, dL + \oint_{L_{20}} \sigma\vec{n}\, dl + \oint_{L_{10}} \sigma\vec{n}\, dl = 0 \quad \text{(divergence theorem)}$$

on same edge

$$\rightarrow \vec{f}_0 = -\oint_{L_{20}} \sigma \vec{n}_{20}\, dl - \oint_{10} \sigma \vec{n}_{10}\, dl = -\sigma\left(\frac{\|\vec{x}_{20}\|}{2}\vec{n}_{20} + \frac{\|\vec{x}_{10}\|}{2}\vec{n}_{10}\right)$$

in 3D

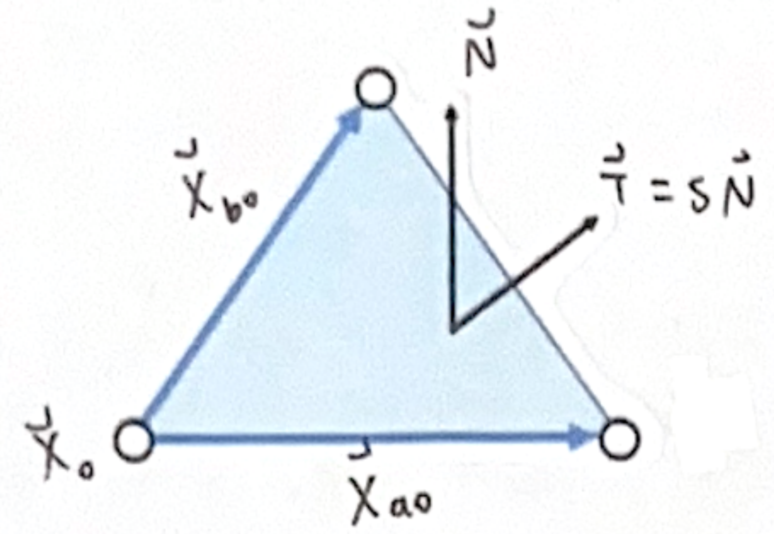
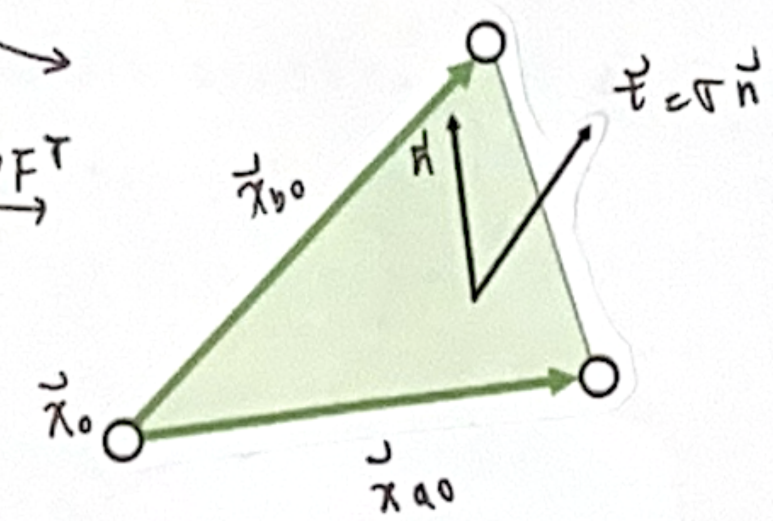


$$\vec{f}_0 = -\oint_\Omega \sigma\vec{n}\, dA = -\sigma\left(\frac{A_{012}}{3}\vec{n}_{012} + \frac{A_{023}}{3}\vec{n}_{023} + \frac{A_{031}}{3}\vec{n}_{031}\right)$$

$$= -\frac{\sigma}{3}\left(\frac{\|\vec{x}_{10}\times\vec{x}_{20}\|}{2}\frac{\vec{x}_{10}\times\vec{x}_{20}}{\|\vec{x}_{10}\times\vec{x}_{20}\|} + \frac{\|\vec{x}_{20}\times\vec{x}_{30}\|}{2}\frac{\vec{x}_{20}\times\vec{x}_{30}}{\|\vec{x}_{20}\times\vec{x}_{30}\|} + \frac{\|\vec{x}_{30}\times\vec{x}_{10}\|}{2}\frac{\vec{x}_{30}\times\vec{x}_{10}}{\|\vec{x}_{30}\times\vec{x}_{10}\|}\right)$$

$$= -\frac{\sigma}{6}\left(\vec{x}_{10}\times\vec{x}_{20} + \vec{x}_{20}\times\vec{x}_{30} + \vec{x}_{30}\times\vec{x}_{10}\right)$$

★ This stress is not that stress

	Interface normal $\vec{N}$ in the refrence state (unformed)	interface normal $\vec{n}$ in the current state (deformed)
Traction in the reference state (unformed)	 In FEM, we define the energy density W in the reference state. Therefore, this stress S is a mapping from normal $\vec{N}$ to the traction $\vec{T}$, both in reference state. Use Second Piola-Kirchhoff Stress as S.	
Traction in the current state (formed)	Use First Piola-kirchhoff Stress as P.	 In FVM, we need σ to convert the normal $\vec{n}$ into $\vec{t}$ for force calculation. Therefore, this stress σ assumes the normal $\vec{n}$ and the traction $\vec{t}$ are in the deform state. Use Cauchy Stress as σ.

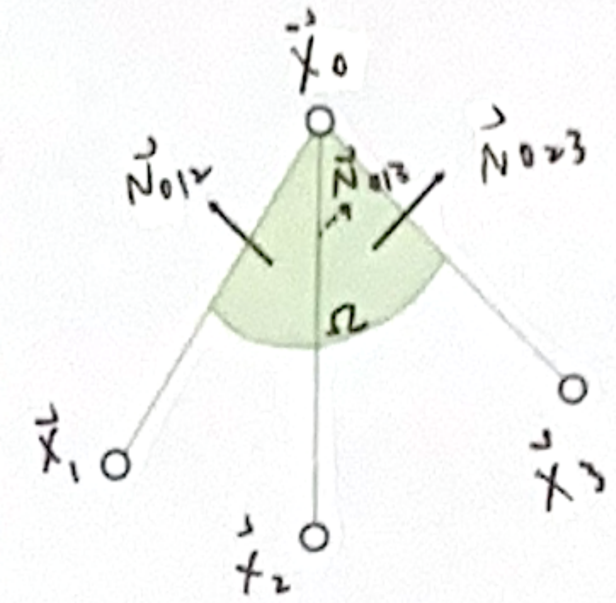
$P = FS$

$\sigma = \det^{-1}(F)\, F S F^T$

$\sigma = \det^{-1}(F)\, P F^T$

▲ Although the use of stress tensor is the same, mapping from the interfece normal to the traction, it can be defined by different configurations.

→ Now we have different stresses, serving the same purpose but in different form.



In the previous previous page, we suggests to calculate force on deformed state. However, we can use reference state instead.

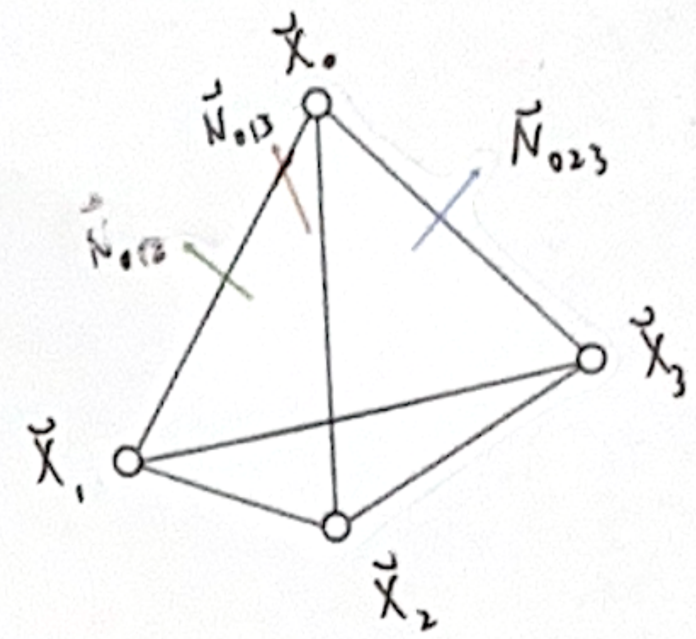
$$\vec{f}_0 = -\frac{\sigma}{6}\left(\vec{x}_{10} \times \vec{x}_{20} + \vec{x}_{20} \times \vec{x}_{30} + \vec{x}_{30} \times \vec{x}_{10}\right)$$

等價!! 因 $\sigma \times$ deformed state $= P \times$ reference state → precompute!

$$= -\frac{P}{6}\left(\vec{X}_{10} \times \vec{X}_{20} + \vec{X}_{20} \times \vec{X}_{30} + \vec{X}_{30} \times \vec{X}_{10}\right)$$

$$= -\frac{FS}{6}\vec{b}_0$$

About $\vec{b}_0$:



$$\vec{X}_{01}^T \vec{b}_0 = \vec{X}_{01}^T\left(\vec{X}_{10} \times \vec{X}_{20} + \vec{X}_{20} \times \vec{X}_{30} + \vec{X}_{30} \times \vec{X}_{10}\right)$$
$$= \vec{X}_{01}^T\left(\vec{X}_{20} \times \vec{X}_{30}\right) = 6\,\mathrm{Vol}$$

$$\vec{X}_{21}^T \vec{b}_0 = \vec{X}_{21}^T\left(\underbrace{\vec{X}_{10} \times \vec{X}_{20}}_{=\vec{X}_{21} \times \vec{X}_{01}} + \underbrace{\vec{X}_{20} \times \vec{X}_{30} + \vec{X}_{30} \times \vec{X}_{10}}_{=\vec{X}_{21} \times \vec{X}_{30}}\right)$$
$$= 0$$

$$\vec{X}_{31}^T \vec{b}_0 = \vec{X}_{31}^T\left(\underbrace{\vec{X}_{10} \times \vec{X}_{20} + \vec{X}_{20} \times \vec{X}_{30}}_{\vec{X}_{13} \times \vec{Y}_{20}} + \underbrace{\vec{Y}_{30} \times \vec{X}_{10}}_{\vec{X}_{31} \times \vec{Y}_{30}}\right)$$
$$= 0$$

$$\rightarrow \left[\vec{X}_{01}\ \vec{X}_{21}\ \vec{X}_{31}\right]^T \vec{b}_0 = \begin{bmatrix} 6\mathrm{Vol} \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\vec{X}_{01}\ \vec{X}_{21}\ \vec{X}_{31}\right]^T \vec{b}_2 = \begin{bmatrix} 0 \\ 6\mathrm{Vol} \\ 0 \end{bmatrix} \quad \rightarrow \left[\vec{b}_0\ \vec{b}_2\ \vec{b}_3\right] = 6\mathrm{Vol}\left[\vec{X}_{01}\ \vec{X}_{21}\ \vec{X}_{31}\right]^{-T}$$

$$\left[\vec{X}_{01}\ \vec{X}_{21}\ \vec{X}_{31}\right]^T \vec{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 6\mathrm{Vol} \end{bmatrix} \quad = \frac{1}{\det\left(\left[\vec{X}_{01}\ \vec{X}_{21}\ \vec{X}_{31}\right]^{-1}\right)}\left[\vec{X}_{01}\ \vec{X}_{21}\ \vec{X}_{31}\right]^{-T}$$

FEM / FVM Framework

$D_m = \left[\vec{X}_{10}\ \vec{X}_{20}\ \vec{X}_{30}\right]$

Deformation gradient — $F = \left[\vec{x}_{10}\ \vec{x}_{20}\ \vec{x}_{30}\right] D_m^{-1}$

Green Strain — $G = \frac{1}{2}\left(F^T F - I\right)$

First PK Stress — $P = F\frac{\partial W}{\partial G}$

Force — $\left[\vec{f}_1\ \vec{f}_2\ \vec{f}_3\right] = -\frac{1}{6\det\left(D_m^{-1}\right)} P D_m^{-T}$

$\vec{f}_0 = -\vec{f}_1 - \vec{f}_2 - \vec{f}_3$

- Hyperelastic Models → 利用能量密度函數,提供了 strain (G) → stress (S) 的映射

★ Isotropic Material (各向同性,在不同方向拉伸或形變,效果一樣)

U和V^T都放外面

Claim that. $P(F) \overset{\text{SVD}}{=} P(UDV^T) = U P(\lambda_0, \lambda_1, \lambda_2) V^T$

deformation gradient

First Piola-Kirchhoff Stress

Principle stretch: the singular value of F

*利用主軸就可以算stress!!

→ In many literature, people parameterize $P(I_c, II_c, III_c)$ by principal invariants:

$C = U^T U$ → right Cauchy-Green deformation tensor

$$\begin{cases} I_c = \text{trace}(C) = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 \\ II_c = \frac{1}{2}(\text{trace}^2(C) - \text{trace}(C^2)) = \lambda_0^2\lambda_1^2 + \lambda_0^2\lambda_2^2 + \lambda_1^2\lambda_2^2 \\ III_c = \det(C^2) = \lambda_0^4 + \lambda_1^4 + \lambda_2^4 \end{cases}$$

Then, the principal stretchs can calculate first Piola-Kirchhoff stress by

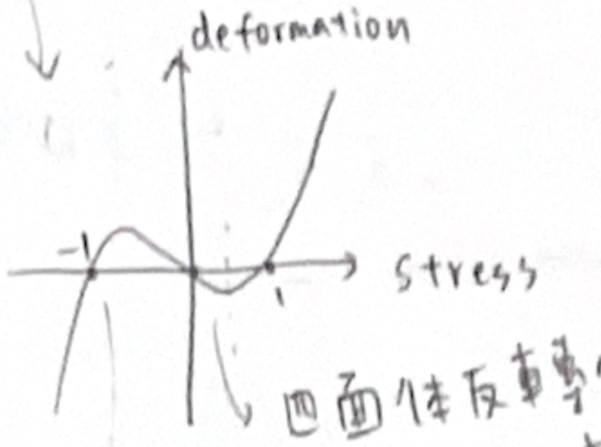
$$P(F) = U P(\lambda_0, \lambda_1, \lambda_2) V^T = U \begin{bmatrix} \frac{\partial W}{\partial \lambda_0} & & \\ & \frac{\partial W}{\partial \lambda_1} & \\ & & \frac{\partial W}{\partial \lambda_2} \end{bmatrix} V^T$$

According to different isotropic model, W has different form.

- Saint Venant-Kirchhoff model (StVK): $W = \frac{S_0}{2}(I_c - 3)^2 + \frac{S_1}{4}(II_c - 2I_c + 3)$
- neo-Hookean model: $W = S_0(III_c^{-\frac{1}{3}} I_c - 3) + S_1(III_c^{-\frac{1}{2}} - 1)$

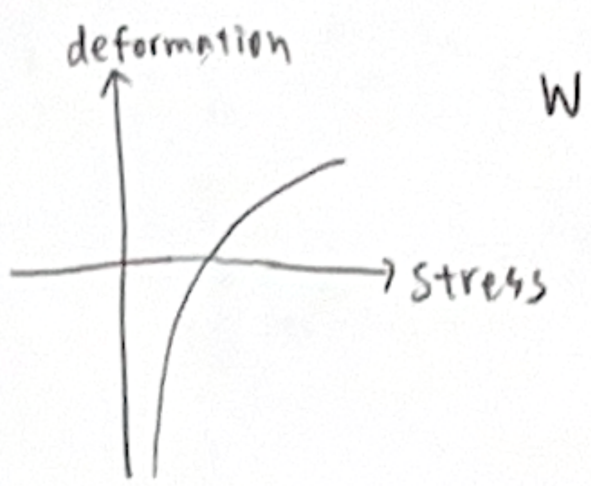
Against shearing 抵抗彈簧形變

Against bulky change (volume change)



四面体反轉後,會在-1點達到平衡狀態

四面体反轉後會翻不回來



FEM/FVM Framework for isotropic model

$D_m = [\vec{X}_{10} \; \vec{X}_{20} \; \vec{X}_{30}]$

deformation gradient — $F = [\vec{x}_{10} \; \vec{x}_{20} \; \vec{x}_{30}] D_m^{-1}$

Principal stretchs Λ — $[U \; \Lambda \; V^T] = \text{svd}(F)$

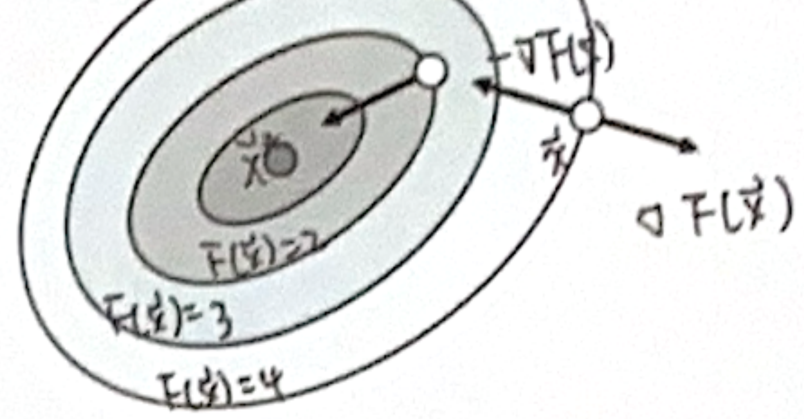
First PK Stress — $P = U \cdot \text{diag}\left(\frac{\partial W}{\partial \lambda_0}, \frac{\partial W}{\partial \lambda_1}, \frac{\partial W}{\partial \lambda_2}\right) V^T$

Force — $[\vec{f}_1 \; \vec{f}_2 \; \vec{f}_3] = -\frac{1}{6\det(D_m^{-1})} P D_m^{-T}$

$\vec{f}_0 = -\vec{f}_1 - \vec{f}_2 - \vec{f}_3$

- Nonlinear Optimization

★ Gradient Descent: a way to solve $\vec{x}^* = \text{argmin}(F(\vec{x}))$



Gradient Descent

Initialize $\vec{x}_0$

for $k = 0 \ldots K$

$\quad \vec{x}^{k+1} = \vec{x}^k - \alpha^k \nabla F(\vec{x}^k)$ (step size)

$\vec{x}^* = \vec{x}^{k+1}$

- How to find the optimal step size?

1. exact line search: (pro) fast convergence, (cons) large overhead, complicated

   try to solve:

   $\alpha = \text{argmin}(F(\vec{x}^k - \alpha \nabla F(\vec{x}^k)))$

2. backtracking line search: (pro) simple, low overhead

   Initialize α

   for $l = 0 \ldots L$

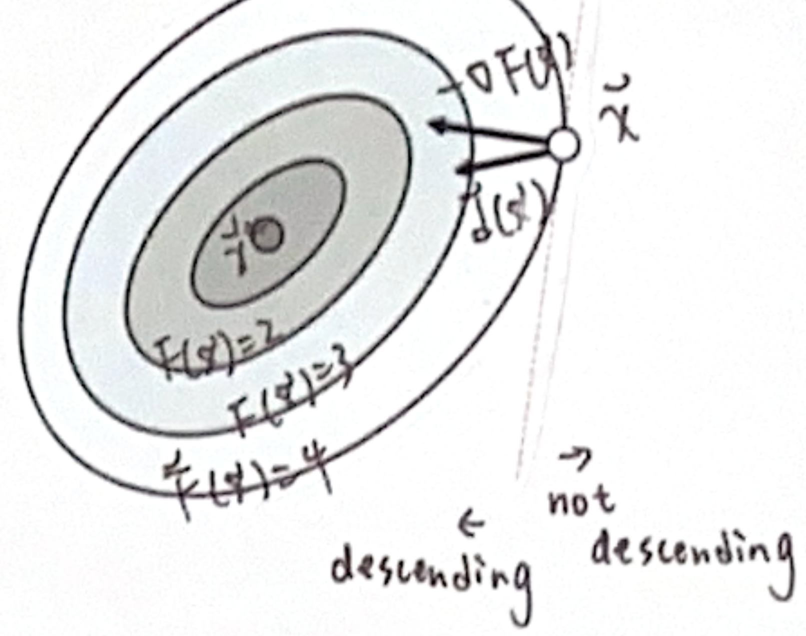
   $\quad$ if $F(\vec{x}^k - \alpha \nabla F(\vec{x}^k)) < F(\vec{x}^k)$

   $\quad\quad$ then break

   $\quad \alpha = \beta\alpha$

   $\beta < 0$, 如果新的 $F(\vec{x}^k - \alpha \nabla F(\vec{x}^k))$ 沒有小於 $F(\vec{x}^k)$,代表走太大步,(已跨過低點),因此需縮小步長

- How to find the optimal descent direction?

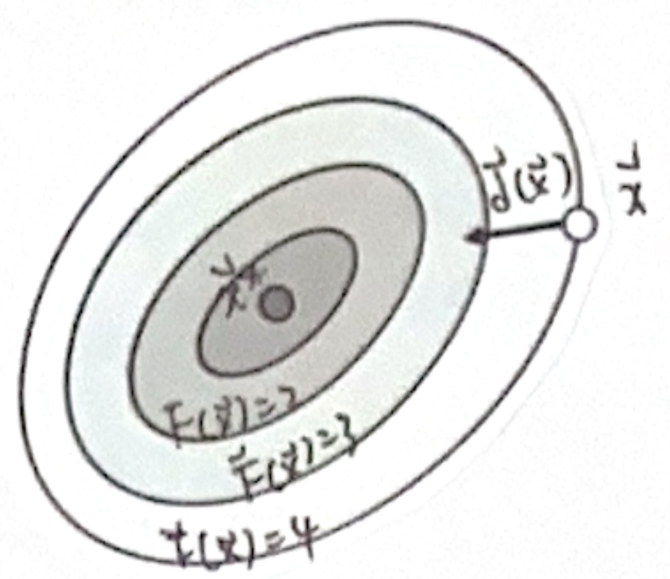


The direction $\vec{d}(\vec{x})$ is descending if a sufficiently small step size α exists for: $F(\vec{x}) > F(\vec{x} + \alpha \vec{d}(\vec{x}))$.

In other words, $-\nabla F(\vec{x}) \cdot \vec{d}(\vec{x}) > 0$

* Descent Methods

With line search, we can use any search direction as long as it's descending: $F(\vec{x}^0) > F(\vec{x}^1) > F(\vec{x}^2) > \dots$



Descent Method

Initialize $\vec{x}^0$

for $k = 0 \dots K$

$\quad \vec{x}^{k+1} = \vec{x}^k + \alpha^k \vec{d}(\vec{x}^k)$

$\vec{x}^* = \vec{x}^{k+1}$

extend

Descent Method with positive definite matrix P

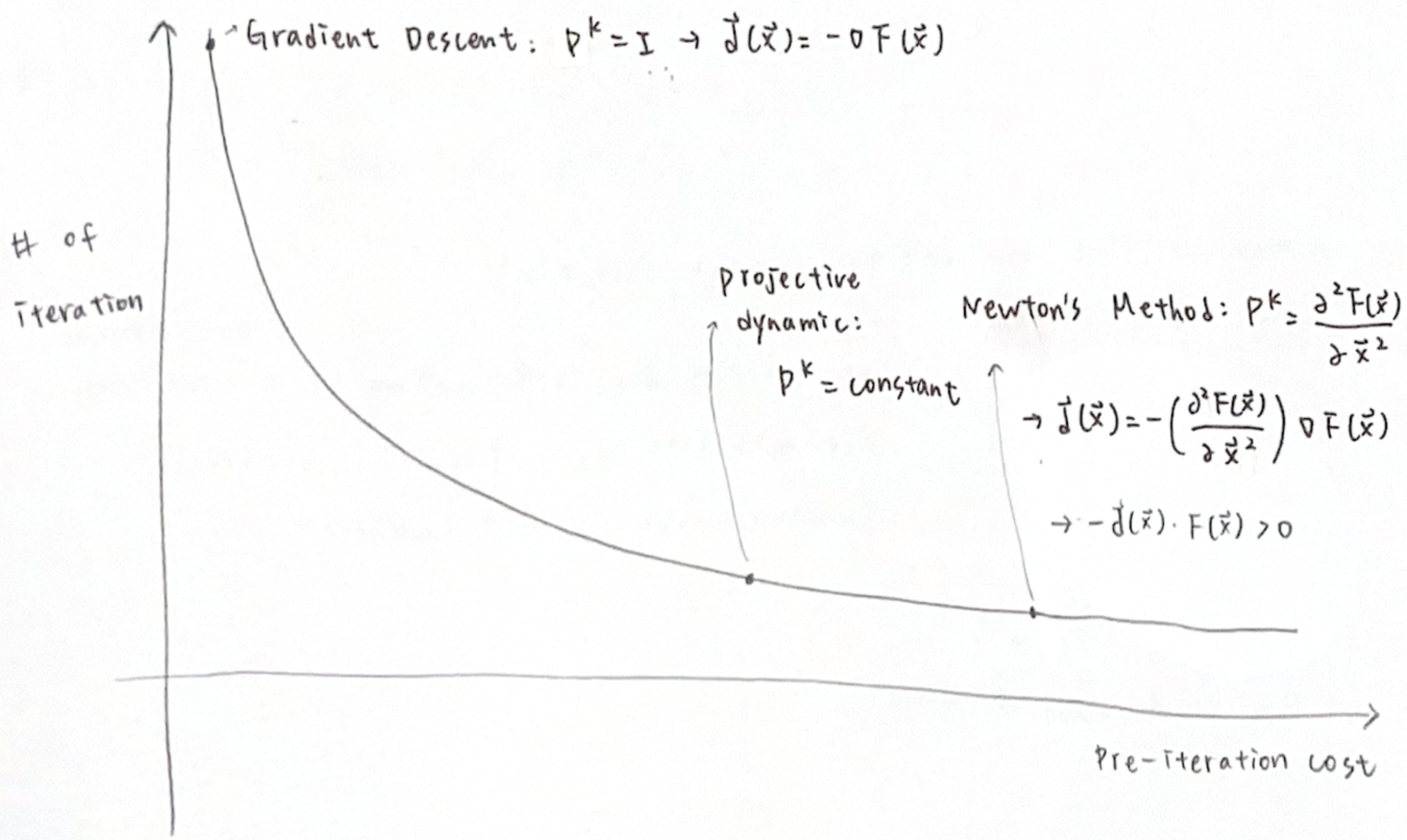
Initialize $\vec{x}^0$

for $k = 0 \dots K$

$\quad \vec{x}^{k+1} = \vec{x}^k - \alpha^k (P^k)^{-1} \nabla F(\vec{x}^k)$

$\vec{x}^* = \vec{x}^{k+1}$

With different P:



▲ Total cost = pre-iteration cost × # of Iteration