Derivation of Gaussian Beam under 2D space and 3D space

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Conditions

Under paraxial condition, the solution is taken as $\psi = u(x, y, z)e^{-ikz}$, where $u(x, y, z)$ is a slowly varying function of z.

$$\left|\frac{\partial u}{\partial z}\right| \ll \left|ku\right|, \left|\frac{\partial^2 u}{\partial z^2}\right| \ll \left|k\frac{\partial u}{\partial z}\right| \ll \left|k^2 u\right|$$

$$\frac{\partial \psi}{\partial z} = \frac{\partial u}{\partial z}e^{-ikz} + (-ik)ue^{-ikz}$$

$$\frac{\partial^2 \psi}{\partial z^2} = \frac{\partial^2 u}{\partial z^2}e^{-ikz} + 2(-ik)\frac{\partial u}{\partial z}e^{-ikz} + (-k^2)ue^{-ikz}$$

Helmholtz equation: $\nabla^2\psi + k^2\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2\psi = 0$

$$\rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (-2ik)\frac{\partial u}{\partial z} = 0$$

Conditions

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- Two dimension:

1. $$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x^2}$$

2. Use Cartesian coordinate system

$$\rightarrow \frac{\partial^2 u}{\partial x^2} - 2ik\frac{\partial u}{\partial z} = 0$$

- Three dimension:

1. $$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \nabla_T^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$$

2. Use Cylindrical coordinate system

$$\rightarrow \nabla_T^2 u - 2ik\frac{\partial u}{\partial z} = 0$$

$$\rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\frac{\partial}{\partial \rho})u + \frac{1}{\rho^2}\frac{\partial^2}{\partial \theta^2}u - 2ik\frac{\partial u}{\partial z} = 0$$

3D Calculation Process

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho})u - 2ik\frac{\partial u}{\partial z} = 0$$

In cylindrical coordinate system with axial symmetry,

$$u = Ae^{-i(p(z)+\frac{k\rho^2}{2q(z)})},$$

where $p(z)$ is the amplitude function which is related to z, and $\frac{k\rho^2}{2q(z)}$ is related to the phase front.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho})u = (-\frac{2ik}{q(z)} - \frac{k^2\rho^2}{q^2(z)})u,\ -2ik\frac{\partial u}{\partial z} = (-2k\frac{\partial P}{\partial z} + \frac{k^2\rho^2}{q^2(z)}\frac{\partial q}{\partial z})u$$

$$\rightarrow (-\frac{2ik}{q(z)} - \frac{k^2\rho^2}{q^2(z)} - 2k\frac{\partial P}{\partial z} + \frac{k^2\rho^2}{q^2(z)}\frac{\partial q}{\partial z})u = 0$$

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$$\begin{cases} \rho^2 : -\dfrac{k^2}{q^2(z)} + \dfrac{k^2}{q^2(z)}\dfrac{\partial q}{\partial z} = 0 \\ 1 : -\dfrac{2ik}{q(z)} - 2k\dfrac{\partial P}{\partial z} = 0 \end{cases} \rightarrow \begin{cases} -1 + \dfrac{\partial q}{\partial z} = 0 \\ \dfrac{i}{q(z)} + \dfrac{\partial P}{\partial z} = 0 \end{cases} \rightarrow \begin{cases} q = z + q_0 \\ p = -i\ln(z + q_0) \end{cases}$$

$$\rightarrow u = Ae^{-\ln(z+q_0) - i\frac{k\rho^2}{2(z+q_0)}}$$

Let $q_0 = -z + is$, where z_0 and s are real.

$$\rightarrow u = Ae^{-\ln(z-z+is) - i\frac{k\rho^2}{2(z-z+is)}} = A\frac{1}{(z-z)+is}e^{-i\frac{k\rho^2}{2}\frac{(z-z_0)-is}{(z-z_0)^2+s^2}}$$

$$= A\frac{1}{(z-z)+is}e^{-\frac{k\rho^2}{2}\frac{s}{(z-z_0)^2+s^2}}e^{-i\frac{k\rho^2}{2}\frac{(z-z_0)}{(z-z_0)^2+s^2}}$$

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$$\frac{1}{(z-z)+is} = \frac{(z-z_0)-is}{(z-z_0)^2+s^2} = \frac{(z-z_0)-is}{s(\frac{(z-z_0)^2}{s^2}+1)} = \frac{\sqrt{\frac{2s}{k}}(\frac{z-z_0}{s}-i)}{\sqrt{\frac{2s}{k}}(\frac{(z-z_0)^2}{s^2}+1)} = \frac{\sqrt{\frac{2s}{k}}i(-1-\frac{z-z_0}{s}i)}{\sqrt{\frac{2s}{k}(\frac{(z-z_0)^2}{s^2}+1)}}$$

$$= \frac{1}{\sqrt{\frac{2s}{k}(\frac{(z-z_0)^2}{s^2}+1)}}\sqrt{\frac{2s}{k}}ie^{i\phi} = \frac{w_0}{w(z)}ie^{i\phi}, \text{ where } \begin{cases} \phi = \tan^{-1}\frac{z-z_0}{s} \\ \frac{1}{w^2(z)} = \frac{ks}{2((z-z_0)^2+s^2)}, w_0 = \sqrt{\frac{2s}{k}} \end{cases}$$

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$$\begin{cases} \dfrac{1}{w^2(z)} = \dfrac{ks}{2((z-z_0)^2+s^2)} \\ \dfrac{1}{R(z)} = \dfrac{z-z_0}{(z-z_0)^2+s^2} \\ \tan\phi = \dfrac{z-z_0}{s} \end{cases}$$

Beam radius

Curvature radius of the phase front

Gouy phase

$$\rightarrow u = A\frac{w_0}{w(z)}e^{\frac{-\rho^2}{w^2(z)}}e^{-i(\frac{k\rho^2}{2R(z)}-\phi)}$$

$$\rightarrow \psi = A\frac{w_0}{w(z)}e^{\frac{-\rho^2}{w^2(z)}}e^{-i(kz+\frac{k\rho^2}{2R(z)}-\phi)}$$

Conditions

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + (-2ik)\frac{\partial u}{\partial z} = 0$$

- Two dimension:

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$$u = Ae^{-i(p(z)+\frac{k\rho^2}{2q(z)})},$$

where $p(z)$ is the amplitude function which is related to z, and $\frac{k\rho^2}{2q(z)}$ is related to the phase front.

In Cartesian coordinate system, simplify $\rho^2 = x^2 + y^2$ as x^2, so

$$u = Ae^{-i(p(z)+\frac{kx^2}{2q(z)})}.$$

2D Calculation Process

$u = Ae^{-i(p(z)+\frac{kx^2}{2q(z)})}$

$$\frac{\partial^2}{\partial x^2}u = (-\frac{ik}{q(z)} - \frac{k^2\rho^2}{q^2(z)})u, -2ik\frac{\partial u}{\partial z} = (-2k\frac{\partial P}{\partial z} + \frac{k^2x^2}{q^2(z)}\frac{\partial q}{\partial z})u$$

$$\rightarrow (-\frac{ik}{q(z)} - \frac{k^2x^2}{q^2(z)} - 2k\frac{\partial P}{\partial z} + \frac{k^2x^2}{q^2(z)}\frac{\partial q}{\partial z})u = 0$$

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$$\begin{cases} x^2 : -\dfrac{k^2}{q^2(z)} + \dfrac{k^2}{q^2(z)}\dfrac{\partial q}{\partial z} = 0 \\ 1 : -\dfrac{ik}{q(z)} - 2k\dfrac{\partial P}{\partial z} = 0 \end{cases} \rightarrow \begin{cases} -1 + \dfrac{\partial q}{\partial z} = 0 \\ \dfrac{i}{2q(z)} + \dfrac{\partial P}{\partial z} = 0 \end{cases} \rightarrow \begin{cases} q = z + q_0 \\ p = -\dfrac{i}{2}\ln(z + q_0) \end{cases}$$

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Let $q_0 = -z + is$, where z_0 and s are real.

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$$= \frac{1}{\sqrt{\frac{2s}{k}(\frac{(z-z_0)^2}{s^2}+1)}}\sqrt{\frac{2s}{k}}ie^{i\phi} = \frac{w_0}{w(z)}ie^{i\phi}, \text{ where } \begin{cases} \phi = \tan^{-1}\frac{z-z_0}{s} \\ \frac{1}{w^2(z)} = \frac{ks}{2((z-z_0)^2+s^2)}, w_0 = \sqrt{\frac{2s}{k}} \end{cases}$$

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$$\begin{cases} \dfrac{1}{w^2(z)} = \dfrac{ks}{2((z-z_0)^2+s^2)} & \text{Beam radius} \\ \dfrac{1}{R(z)} = \dfrac{z-z_0}{(z-z_0)^2+s^2} & \text{Curvature radius of the phase front} \\ \tan\phi = \dfrac{z-z_0}{s} & \text{Gouy phase} \end{cases}$$

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2D v.s. 3D

$$\psi_{2D} = A\sqrt{\frac{w_0}{w(z)}}e^{\frac{-x^2}{w^2(z)}}e^{-i(kz+\frac{kx^2}{2R(z)}-\frac{1}{2}\phi)}$$

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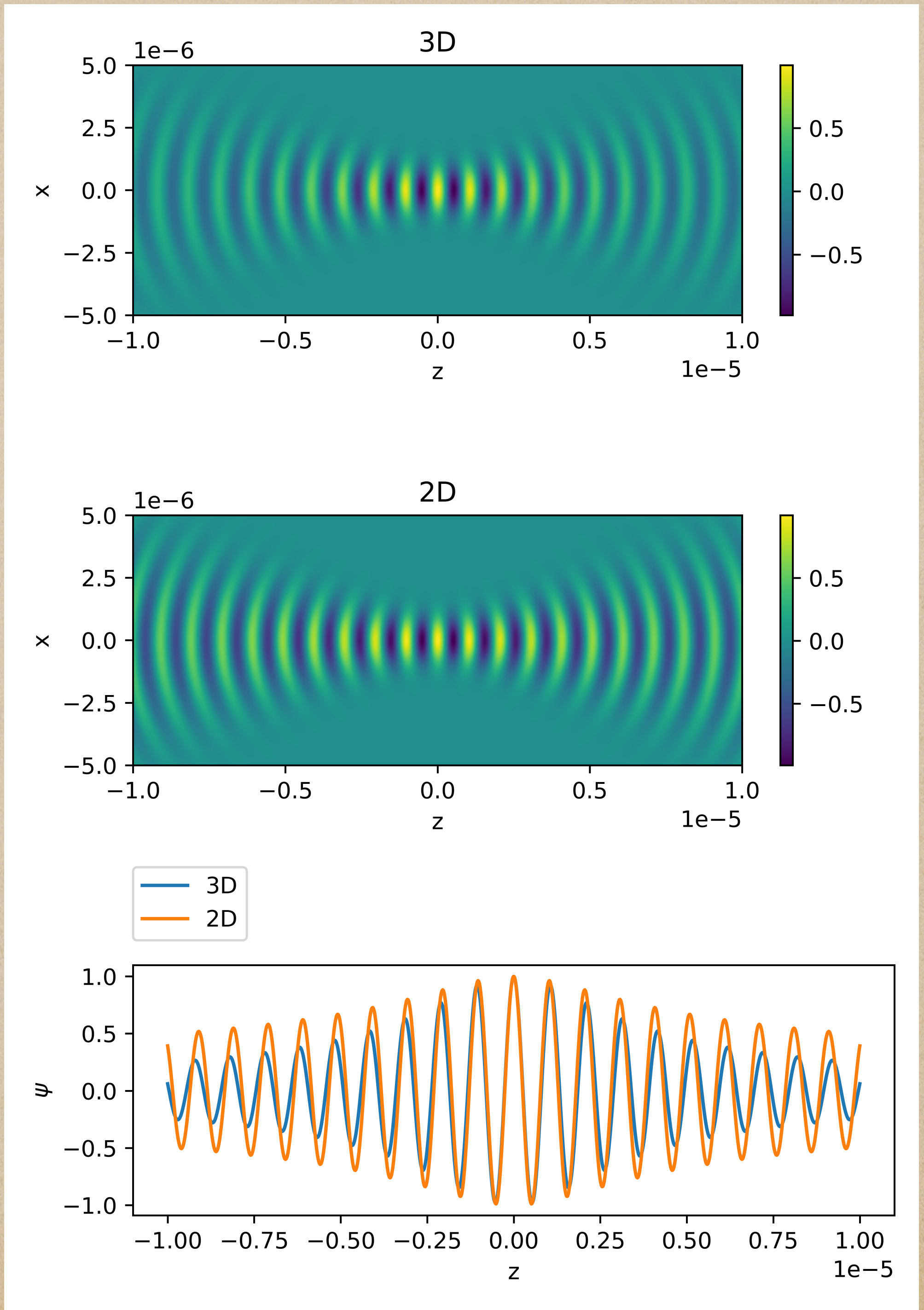

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